UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH3801**
MODULE NAME	:	**Logic**
DATE	:	**18-May-07**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2006/07-MATH3801A-001-EXAM-27

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of *primitive recursive function, partial recursive function* and *recursive function.*

   (b) Give the definitions of *Turing machine, Turing program* and *Turing computable partial function.*

   (c) Let ϕ_n denotes the n^{th} partial recursive function in some numbering and let $K = \{n \in \omega : \phi_n(n) \text{ converges}\}$. Show that K is recursively enumerable, but not recursive.

   (d) A function $f : \omega \to \omega$ is increasing if $f(n) < f(n+1)$ for all $n \in \omega$. Show that an infinite set A is recursive if and only if A is the range of an increasing recursive function.

2. (a) Give the definition of a *propositional language* L and the set of L*-formulas.*

   (b) State what it means for an L-formula to be in *disjunctive normal form*, and use the truth table method to put $(p \,\&\, \neg q) \Rightarrow r$ in disjunctive normal form.

   (c) Give the definition of the *formation tree* of an L-formula and give the unique formation tree for the L-formula

$$(((p \vee q) \,\&\, (q \wedge r)) \Rightarrow (p \wedge r)) \Rightarrow \neg(q \& s)$$

3. (a) In the Predicate Calculus a language L is a set of predicates with each predicate P of L being n-ary for some $n \in \omega$. Define the *variables, quantifiers, L-U-formulas* for a set U, the *degree* of formulas, and *L-U-sentences.*

   (b) Let U be a nonempty set and S a set of unsigned L-U-sentences. Define what it means for S to be *open* and to be *U-replete.* State Hintikka's Lemma for the Predicate Calculus.

   (c) Let τ_0 be a finite tableau. Prove that we can extend τ_0 to a (possibly infinite) tableau τ with the following properties: every closed path of τ is finite, and every open path of τ is V-replete, where $V = \{a_1, \ldots, a_n, \ldots\}$ is a set of parameters.

   (d) State and prove the Completeness Theorem for the Predicate Calculus.

4. Let $\mathcal{L} = (+, -, \cdot, 0, 1, <, =)$ be the language of ordered rings and $(\mathbb{R}, +_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, <_{\mathbb{R}}, =_{\mathbb{R}})$ the ordered field of real numbers.

   (a) Give the definitions of an *effective subset* A of $\mathbb{R}^n$ and the definition of an *effective function* $f : B \to \mathbb{R}$, $B \subset \mathbb{R}^n$.

   (b) Prove that if $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}$ and $\hat{a} = a_0, \ldots, a_n$, then there are $n+1$ effective functions $\eta_1(\hat{a}) < \ldots < \eta_n(\hat{a})$, and $k = k(\hat{a})$ such that $\eta_1(\hat{a}) < \cdots < \eta_k(\hat{a})$ are all of the real roots of $p(x)$.

5. (a) Give the definition of an *arithmetical definable* k-place partial function for the structure $(\mathbb{N}, +, \cdot, 0, 1, =)$.

   (b) For each formula F in the language of arithmetic let $\sharp(F)$, be the Gödel number of F. If $\mathcal{F}$ is any collection of formulas, $\sharp\mathcal{F}$ denotes $\{\sharp(F) : F \in \mathcal{F}\}$. Let $TrueSnt = \sharp\{\text{all true sentences}\}$. Show that every arithmetically definable set $A \subset \mathbb{N}$ is reducible to $TrueSnt$. Conclude that there is no algorithm to compute the characteristic function of $TrueSnt$.